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On the out-domination and in-domination numbers of a digraph

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Dedicated to the memory of the late great John von Neumann who discovered domination in digraphs

Abstract

An out-domination set of a digraph D is a set S of vertices of D such that every vertex of $D-S$ is adjacent from some vertex of S. The minimum cardinality of an out-domination set of D is the out-domination number $\gamma^+(D)$. The in-domination number $\gamma^-(D)$ is defined analogously. It is shown that for every digraph D of order n with no isolates, $\gamma^+(D)+\gamma^-(D)\leqslant 4n/3$. Furthermore, the digraphs D for which equality holds are characterized. Other inequalities are also derived.

Keywords: Out-domination; In-domination; Digraph

1. Introduction

Domination in graphs has been a subject of much study in recent years. Indeed, the book by Haynes et al. [6] is entirely devoted to this area. A survey of domination in digraphs is given in [2]. Our primary objective is to study the sum of the out-domination and in-domination numbers of a digraph.

A *digraph* D is a nonempty finite set V of elements called *vertices* together with a collection E of ordered pairs of distinct vertices called *arcs*. An *oriented graph* is a digraph with no symmetric pairs of arcs. Equivalently, an oriented graph is a digraph that can be obtained from a graph G by assigning one orientation to each edge of G.

An *out-domination set* of a digraph D is a set S of vertices of D such that every vertex of $D-S$ is adjacent *from* some vertex of S. The minimum cardinality of an

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out-domination set for D is the *out-domination number* $\gamma^+(D)$ of D. The *in-domination number* $\gamma^-(D)$ is defined as expected. Following the books [1,3–5], we denote the *converse* of D by D', obtained by reversing the direction of every arc of D. Then clearly, $\gamma^-(D')=\gamma^+(D)$ for every digraph D.

Some observations are now in order. For subdigraphs D_1 and D_2 of a digraph D with $V(D_1)\cup V(D_2)=V(D)$,

$$\gamma^+(D)\leqslant\gamma^+(D_1)+\gamma^+(D_2) \quad\text{and}\quad \gamma^-(D)\leqslant\gamma^-(D_1)+\gamma^-(D_2). \tag{1}$$

Also, a vertex with in-degree 0 belongs to every out-dominating set and a vertex with out-degree 0 belongs to every in-dominating set.

2. The sum of the out-domination and in-domination numbers of digraphs

Our main theorem is an inequality which will first be proved in a special case. The *degree* $\deg v$ of a vertex v in a digraph is the sum of its in-degree and out-degree, that is, $\deg v=\mathrm{id}\, v+\mathrm{od}\, v$. A vertex v is an *end-vertex* if $\deg v=1$. The directed path of order n is denoted by $\vec{P}_n$.

Lemma 2.1. *Let S be an oriented star of order $n\geqslant 2$. Then $\gamma^+(S)+\gamma^-(S)\leqslant 4n/3$ with equality if and only if $S=\vec{P}_3$.*

Proof. Let v be either vertex of S if $n=2$, and otherwise the central vertex of S, in which case $\deg v\geqslant 2$. Let $\mathrm{od}\ v = a$ and $\mathrm{id}\ v = b$. Then, by the final remark of the previous section,

$$\gamma^+(S)+\gamma^-(S)=\begin{cases} b+1=n & \text{if } a=0,\\ a+1=n & \text{if } b=0,\\ (b+1)+(a+1)=n+1 & \text{if } a,b\geqslant 1.\end{cases}$$

Now $\gamma^+(S)+\gamma^-(S)\leqslant 4n/3$ in all cases, with equality if and only if $a=b=1$ and $n=3$, when $S=\vec{P}_3$. □

This result can be extended to arbitrary connected digraphs. A digraph is *connected* if it is weakly connected, that is, if its underlying graph is connected.

Theorem 2.2. *For every connected digraph D of order $n\geqslant 2$,*

$$\gamma^+(D)+\gamma^-(D)\leqslant\frac{4n}{3}.$$

Proof. It suffices to verify the result for oriented trees since if T is a spanning oriented tree of a connected digraph D, then $\gamma^+(D)\leqslant\gamma^+(T)$ and $\gamma^-(D)\leqslant\gamma^-(T)$. Thus if $\gamma^+(T)+\gamma^-(T)\leqslant 4n/3$, then $\gamma^+(D)+\gamma^-(D)\leqslant 4n/3$.

We now show by induction on n that $\gamma^+(T)+\gamma^-(T)\leqslant 4n/3$ for every oriented tree of order $n\geqslant 2$. For $n=2$, the result follows from Lemma 2.1. Assume then for all oriented trees T' of order k, where $2\leqslant k<n$ and $n\geqslant 3$, that $\gamma^+(T')+\gamma^-(T')\leqslant 4k/3$, and let T be an oriented tree of order n. If T is an oriented star, then it follows by Lemma 2.1 that $\gamma^+(T)+\gamma^-(T)\leqslant 4n/3$. Hence, we may assume that T is not an oriented star. Therefore, T contains an arc whose deletion results in two nontrivial oriented trees T_1 and T_2. It now follows from the induction hypothesis that $\gamma^+(T_i)+\gamma^-(T_i)\leqslant 4|V(T_i)|/3$ for $i=1,2$ and from (1) that

$$\gamma^+(T)+\gamma^-(T)\leqslant 4|V(T_1)|/3+4|V(T_2)|/3=4n/3. \qquad \square$$

There is an immediate consequence of Theorem 2.2.

Corollary 2.3. *If D is a digraph of order n with no isolates, then $\gamma^+(D)+\gamma^-(D)\leqslant 4n/3$.*

We now present a characterization of those nontrivial connected digraphs of order n for which the equality $\gamma^+(D)+\gamma^-(D)=4n/3$ holds. The directed cycle of order $n\geqslant 3$ is denoted by $\vec{C}_n$.

Theorem 2.4. *Let D be a connected digraph of order n. Then $\gamma^+(D)+\gamma^-(D)=4n/3$ if and only if (1) $D=\vec{C}_3$, (2) $D=\vec{P}_3$, or (3) every vertex of D is either an end-vertex, or is adjacent to exactly one end-vertex and adjacent from exactly one end-vertex.*

Proof. When $D=\vec{C}_3$ or $D=\vec{P}_3$, $\gamma^+(D)+\gamma^-(D)=4$, which is $4n/3$ for $n=3$. Consider next a connected digraph D of order $n\geqslant 2$ possessing property (3), and let D have k vertices that are not end-vertices. Necessarily, D has $2k$ end-vertices. Then $\gamma^+(D)=\gamma^-(D)=2k$; so $\gamma^+(D)+\gamma^-(D)=4k=4n/3$.

Conversely, let D be a connected digraph of order $n\geqslant 2$ for which $\gamma^+(D)+\gamma^-(D)=4n/3$. We show that (1), (2), or (3) holds. By hypothesis, n is a multiple of 3. For $n=3$, we see that $D=\vec{C}_3$ or $D=\vec{P}_3$. Hence we may take n as a multiple of 3 and $n\geqslant 6$.

First, we observe that if D_1 is a subdigraph of D and $D_2=D-V(D_1)$, and if neither D_1 nor D_2 has isolates, then (1) and Corollary 2.3 give

$$\begin{aligned}\gamma^+(D)+\gamma^-(D) &\leqslant \gamma^+(D_1)+\gamma^-(D_1)+\gamma^+(D_2)+\gamma^-(D_2)\\ &\leqslant 4|V(D_1)|/3+4|V(D_2)|/3=4n/3. \end{aligned} \tag{2}$$

Since $\gamma^+(D)+\gamma^-(D)=4n/3$, equality holds throughout (2), so, in particular, $\gamma^+(D_1)+\gamma^-(D_1)=4|V(D_1)|/3$. Consequently, $|V(D_1)|$ is a multiple of 3 and, in addition, by Lemma 2.1,

$$\text{if } D_1 \text{ is a star, then } D_1 = \vec{P}_3. \tag{3}$$

We have therefore established the following fact.

Fact. *If D_1 is a subdigraph of D without isolates whose order is not a multiple of* 3, *then $D - V(D_1)$ contains isolates.*

Now let v be a vertex of D that is not an end-vertex. It remains to show that v is adjacent to exactly one end-vertex and adjacent from exactly one end-vertex. We consider two cases.

Case 1: *v is adjacent with end-vertices.* Let D_1 be the oriented star induced by v and the end-vertices with which it is adjacent. Then the components of $D - V(D_1)$ are nontrivial, and so $D_1 = \vec{P}_3$ by (3). This is the desired result.

Case 2: *v is not adjacent with end-vertices.* Let u be a vertex of D adjacent with v. Taking D_1 to be the subdigraph $\langle\{u,v\}\rangle$ induced by u and v, we see, by the Fact, that $D - u - v$ contains isolates. Since v is not adjacent with end-vertices, either u and v are mutually adjacent with a vertex of degree 2, or u is adjacent with end-vertices.

First, assume that u and v are mutually adjacent with a vertex x of degree 2. Letting $D_1 = \langle\{v,x\}\rangle$, we have, by the Fact, that $D - v - x$ contains isolates and so $\deg u = 2$. Since D is a connected digraph of order $n \geqslant 6$, it follows that $\deg v > 2$. Hence $D - u - x$ contains no isolates. Choosing $D_1 = \langle\{u,x\}\rangle$, we obtain a contradiction to the Fact. Therefore, u and v are not mutually adjacent with a vertex of degree 2. Necessarily, then, u is adjacent with end-vertices. Applying Case 1 to u, we see that there are exactly two such end-vertices, say u_1 and u_2. Consequently, $D - \{v,u,u_1,u_2\}$ has no isolates. However, taking $D_1 = \langle\{v,u,u_1,u_2\}\rangle$ once again produces a contradiction to the Fact. Thus Case 2 cannot occur and the proof is complete. □

We have already mentioned that $\gamma^-(D') = \gamma^+(D)$ for every digraph D, where D' denotes the converse of D; and of course, $\gamma^+(D') = \gamma^-(D)$. Now, by Corollary 2.3, $\gamma^+(D) + \gamma^-(D) \leqslant 4n/3$ for every digraph D of order $n \geqslant 2$ having no isolates. Since the geometric mean of two positive real numbers never exceeds their arithmetic mean, it follows that $\gamma^+(D)\gamma^-(D) \leqslant 4n^2/9$ for every digraph D of order $n \geqslant 2$ having no isolates. That this bound is sharp is a consequence of Theorem 2.4.

As $\gamma^+(D) \geqslant 1$ and $\gamma^-(D) \geqslant 1$ for every digraph D, we have $\gamma^+(D)\gamma^-(D) \geqslant 1$ and $\gamma^+(D) + \gamma^-(D) \geqslant 2$. Now let F be a digraph of order $n \geqslant 2$ containing distinct vertices u and v such that u is adjacent *to* all other vertices of F and v is adjacent *from* all other vertices of F. Hence $\gamma^+(F) = \gamma^-(F) = 1$ and the lower bounds for $\gamma^+(D)\gamma^-(D)$ and $\gamma^+(D) + \gamma^-(D)$ mentioned above are sharp. This provides us with a proof of the following result, which can be interpreted as a modified Nordhaus–Gaddum theorem for digraphs, where the converse of a digraph replaces the complement of a graph.

Theorem 2.5. *For every digraph D of order $n \geqslant 2$ with no isolates, the following bounds are sharp*:

$$2 \leqslant \gamma^{+}(D) + \gamma^{-}(D') \leqslant \frac{4n}{3},$$

$$1 \leqslant \gamma^{+}(D)\gamma^{-}(D) \leqslant \frac{4n^2}{9}.$$

The proof of Theorem 2.4 also gives the following result.

Corollary 2.6. *Let D be a connected digraph of order $n \geqslant 3$. Then $\gamma^{+}(D)\gamma^{-}(D) = 4n^2/9$ if and only if* (1) $D = \vec{C}_3$, (2) $D = \vec{P}_3$, *or* (3) *every vertex of D is either an end-vertex, or is adjacent to exactly one end-vertex and adjacent from exactly one end-vertex.*

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